20 Years of Negami's Planar Cover Conjecture

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September 5, 2008

Abstract

In 1988, Seiya Negami published a conjecture stating that a graph G has a finite planar cover (i.e. a homomorphism from some planar graph onto G which maps the vertex neighbourhoods bijectively) if and only if G embeds in the projective plane. Though the "if" direction is easy, and some supporting weaker statements have been shown by him, the conjecture is still open, after more than 20 years of intensive investigation. We review the (quite significant) progress made so far in solving Negami's conjecture, and propose possible promising directions of future research.

1 Planar covers

We deal only with finite undirected graphs, and assume that the reader is familiar with basic terms of topological graph theory, e.g. with [14].

We start with a precise formal definition of a cover which we then relax to a less formal and more usable variant.

Definition. A graph H is a *cover* of a graph G if there exist a pair of onto mappings (φ, ψ), $\varphi : V(H) \to V(G)$, $\psi : E(H) \to E(G)$, called a (cover) *projection*, such that ψ maps the edges incident with each vertex v in H bijectively onto the edges incident with $\varphi(v)$ in G.

In particular, for $e = uv$ in H, the edge $\psi(e)$ in G has ends $\varphi(u), \varphi(v)$. Thus, for simple graphs, it is enough to specify the vertex projection φ that maps the neighbors of each vertex v in H bijectively onto the neighbors of $\varphi(v)$ in G (a traditional approach). If G' is a subgraph of G, then the

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graph H' with the vertex set $\varphi^{-1}(V(G'))$ and the edge set $\psi^{-1}(E(G'))$ is called a *lifting of G' into H*.

To illustrate the concept of a cover, we present several basic properties:

- Degree preservation; $d_H(v) = d_G(\varphi(v))$ for each vertex $v \in V(H)$.
- Lifting of a path P of G into H consists of a collection of disjoint paths isomorphic to P. Hence, if G is connected, then $|\varphi^{-1}(v)| = k$ is the same number for all $v \in V(G)$. We then speak about a k-fold cover.
- Lifting of a cycle C_n of G consists of a collection of disjoint cycles whose lengths are divisible by n.
- Any graph embedded in the projective plane has a double cover which is planar, via the universal covering map from the sphere onto the projective plane.
- If G has a cover which is planar, then so does every minor of G.
- Let e be an edge of G between two neighbours of some cubic vertex. If $G - e$ has a cover which is planar, then so does G. Therefore, if G has a planar cover and G' is obtained from G by $Y\Delta$-transformations (replacing a cubic vertex with a triangle on the neighbours), then G' has a planar cover.

Interest in graphs having a cover which is planar has been raised by Negami [15] in relation to enumeration of distinct projective embeddings of 3-connected graphs. Interestingly, a very similar concept has been introduced and studied independently at the same time by Fellows [4], cf. also [5]. (Fellows' concept of planar emulators has been later considered also by Kitakubo [13] under the name of branched planar covers.) One of the early results on planar covering is an immediate corollary of [15]:

Theorem 1 (Negami, 1986) *A connected graph has a double planar cover if and only if it embeds in the projective plane.*

A natural extension of this result is provided with the concept of regular covers [16]. A cover $\varphi : V(H) \to V(G)$ is *regular* if there is a subgroup A of the automorphism group of H such that $\varphi(u) = \varphi(v)$ for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some automorphism $\tau \in A$.

Theorem 2 (Negami, 1988) *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

It is worth to note, though Negami's proofs of the above two results are based on topological arguments and the fact that the universal covering space of the projective plane is the sphere, we can provide alternative clean combinatorial arguments (yet unpublished) for both the theorems.

We informally sketch the combinatorial idea for Theorem 1: Let a double cover of G be a 3-connected plane (embedded) graph H (handling of non-3-connected cases is rather technical but straightforward, cf. also Negami's arguments [15, 16]). If T is any spanning tree of G, then T lifts into two trees T' and T'' in H, and the mapping "exchanging" T' with T'' is an automorphism of H. By uniqueness of the plane embedding of 3-connected H, the rotation scheme of vertices of T' in H hence must be the same as that of T'' in H. Consequently, the edges "between" T' and T'' in H have the same cyclic orderings when viewed from T' as from T''. If these two orderings are of opposite orientation, then we easily extend T' into a plane embedding of G, and with same orientation we analogously get an embedding of G with one crosscap. Furthermore, with little extension at the end, the same idea can be used to prove also Theorem 2.

2 Negami's conjecture

Theorem 2 suggests the following immediate generalization [16]:

Conjecture 3 (Negami, 1988) *A connected graph has a finite planar cover if and only if it embeds in the projective plane.*

Although the two statements sound very similar, the real jump in their difficulty seems enormous. No proof ideas of Theorem 2 reasonably extend towards solving Conjecture 3; the main reason being lack of "regularity", or symmetry, in the cover graph. Consequently, despite a long chain of promising partial results (and one finalizing announcement in 2004 with no written proof yet), Conjecture 3 is still open in 2008.

All the mentioned partial results follow a simple scheme developed at the beginning by Archdeacon and Negami: Easily, if a graph embeds in the projective plane, then it has a double planar cover (Theorem 1). Conversely, there is a known list [6, 1] of all 35 forbidden minors for the graphs embeddable in the projective plane, see them in the Appendix. So if a connected graph G does not embed in the projective plane, then G has F, one of the connected 32 graphs of that list, as a minor. If we can prove that F has no finite planar cover, then neither has G by the above observation. Furthermore, as observed by Archdeacon, the list can be further shortened using $Y\Delta$-transformations.

Though the problem now sounds as a finite check of (at most) 32 graphs, we remind the reader that even looking for a planar cover of just one graph does not seem to be a finite task so far.

Disjoint k-graphs

Actually, quite large portion of the 32 graphs can be covered with a simple general argument; it is enough to know that a graph contains "two disjoint k-graphs" to argue that it has no finite planar cover. The rather complicated notion of k-graphs was introduced already in [6] and we refer the reader to e.g. [9, Section 2.3] for a precise formulation.

The mentioned argument, discovered by Negami [17] and Archdeacon [unpublished], gives the following result.

Theorem 4 (Negami / Archdeacon, 1988)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar cover. (See the Appendix for notation.)*

We informally sketch its idea on a particular case of $K_5 \cdot K_5$, but a full generalization is quite straightforward. Let c be the degree-8 vertex of $K_5 \cdot K_5$, and A_4 and B_4 be the two (isomorphic to K_4) components of $K_5 \cdot K_5 - c$. Consider a finite plane-embedded cover H of $K_5 \cdot K_5$, and assume, up to symmetry, that it is a component H_4 of the lifting of A_4 into H that contains no part of lifting of B_4 in its internal faces. However, H_4 as a cubic graph cannot be outerplanar, and hence some internal vertex x of H_4 is adjacent to some y in the lift of c into H, and this y must be adjacent to vertices in the lifting of B_4, a contradiction. Hence $K_5 \cdot K_5$ has no finite planar cover.

Two discharging arguments

Discharging is a proof method developed mainly along the Four colour problem. The method simply applies Euler's formula in a clever way.

A very easy discharging argument shows that the graph $K_{3,5}$ cannot have a finite planar cover. Though this claim is first attributed to Fellows, it does not occur in [4]. A short published proof can be found, e.g., in [12].

Theorem 5 (1988, 1993) *The graph* $K_{3,5}$ *has no finite planar cover.*

In a sketch again, suppose that a (bipartite) graph H was a finite cover of $K_{3,5}$ embedded in the plane. We assign charge of $3(4 - d_H(v))$ to every vertex v, and of $3(4 - len(\phi))$ to every face ϕ of H. By Euler's formula, the total charge of H is $12 \cdot 2 > 0$. Then every 3-vertex of H sends its charge equally 1 to each neighbour. So every 5-vertex x of H now has charge of $-3+5 = 2$. That charge is subsequently sent from x to any incident ≥ 6-face of H. Say x is a lift of a vertex a of the smaller vertex part $\{a, b, c\}$ of $K_{3,5}$, and so the second neighbourhood of x in H contains only vertices lifting from the other two b, c by definition of a cover. Hence it cannot happen that all the incident faces of x in H are quadrilaterals, as lifts of b and of c cannot alternate around the dual 5-cycle of x. Therefore, all vertices of H end up with nonnegative charge, and so do all the faces as one may easily count. So, where has all the positive charge gone? This contradiction concludes Theorem 5.

Another, significantly more involved discharging argument has been found several years later by the author [7] for the case of $K_{4,4} - e$.

Theorem 6 (PH, 1998) *The graph $K_{4,4} - e$ has no finite planar cover.*

A noticeable feature of the proof is that discharging is applied not to the supposed planar cover itself, but to a special simplification of it. That seems the right way to go, as a successful case of $\mathcal{E}_2$ in Theorem 9 also shows.

Structural approach

Yet another approach to prove nonexistence of a planar cover was discovered by Archdeacon already in 1988, but the proof had not been published until much later in [2].

Theorem 7 (Archdeacon, 1988, 2002)
The graphs $K_7 - C_4$ and $K_{4,5} - 4K_2$ have no finite planar covers.

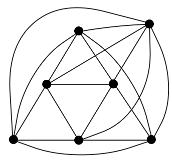
Here the proof cannot be easily sketched, and so we mention only that it looks for a short "necklace" of interconnected 4-cycles in the supposed cover, and then finds a way the necklace can be made even shorter, arriving at a contradiction. One can say that this idea is a wide generalization of the "disjoint k-graph" argument of Theorem 4.

Interestingly, the exactly same proof was rediscovered (independently) by Thomas and the author 10 years later, see [9], and subsequently generalized by the author to cover also the case of $\mathcal{C}_4$ in Theorem 9 in the next section. The particularly nice feature of this generalization is that its proof directly constructs from the shortest necklace a projective embedding of the covered graph, instead of deriving an artificial contradiction, cf. also Theorems 1,2.

3 The bad: $K_{1,2,2,2}$ and relatives

After all, putting together Theorems 4, 5, 6, and 7, and applying $Y\Delta$-transformations to the graphs $\mathcal{D}_3, \mathcal{E}_5, \mathcal{F}_1, \mathcal{B}_7, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2$, leaves only one following case to be resolved.

Corollary 8 (1998) *If the graph $K_{1,2,2,2}$ (the octahedron with an extra vertex) has no finite planar cover, then Conjecture 3 holds true.*



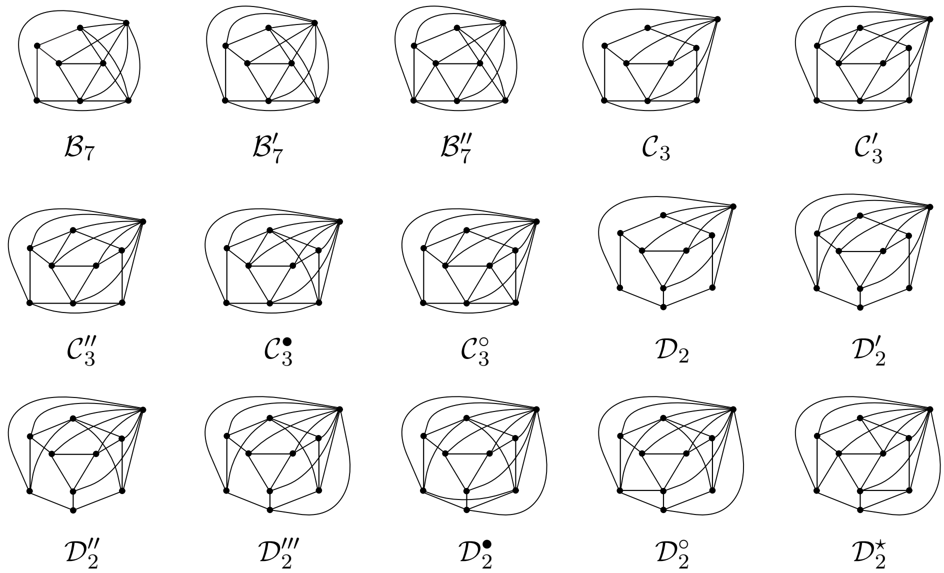
It might seem that Corollary 8 "almost solves" Negami's conjecture, but the opposite appears true, since even more than 10 years later the conjecture is still wide open. Notice that there are other graphs on the list of projective obstructions which are unsolved yet. Namely, the graph $\mathcal{C}_4$ reduces via $\mathcal{B}_7$ to $K_{1,2,2,2}$, and so does the graph $\mathcal{E}_2$ via $\mathcal{D}_2$, $\mathcal{C}_3$ and $\mathcal{B}_7$. Hence in the situation when we are not able to attack the final case of $K_{1,2,2,2}$ directly, it might perhaps help to "train our muscles" on some of the easier cases. Such a strategy led to the following new results [9, 10]:

Theorem 9 (PH, 1999 and 2001) *The graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ (otherwise both $Y\Delta$-transformed to $K_{1,2,2,2}$) have no finite planar covers.*

Despite the graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ are relatives in a "family of $K_{1,2,2,2}$", the proofs for each one of them are completely different and incomparable. While an involved discharging argument is applied in the case of $\mathcal{E}_2$, the other case of $\mathcal{C}_4$ is covered by a generalization of the necklace argument from Theorem 7. Unfortunately, neither of these arguments can be directly generalized to any other of the missing cases! So, we suggest that the right way to attack the case of $K_{1,2,2,2}$ is to find a suitable *common generalization of the structural and discharging approaches* of Section 2.

One may proceed even further in the direction of Theorem 9, and ask for which of all graphs, to our current knowledge, Conjecture 3 might possibly fail. That direction has been taken by Thomas and the author in [9] and [11]. A *planar expansion* of a graph G is a graph which results from G by adding a planar graph sharing one vertex with G, or by replacing an edge or a cubic vertex with a planar graph with its attachments on the outer face. We refer to [9, Section 6.1] for a formal description.

Theorem 10 (PH and Thomas, 1999 and 2004)
Let Π be the set of $K_{1,2,2,2}$ and the 15 graphs listed below. If a connected graph G has a finite planar cover but no projective embedding, then G is a planar expansion of some graph from Π.



Furthermore, we can order those possible counterexamples from Π according to their difficulty, and then perhaps apply our "muscle-training" strategy with respect to this ordering. Let us write $G \xrightarrow[\text{NC}]{} H$ to mean that "if G has no planar cover, then neither has H". We easily get:

$$K_{1,2,2,2} \xrightarrow[\text{NC}]{} \mathcal{B}_7 \xrightarrow[\text{NC}]{} \mathcal{B}_7' \xrightarrow[\text{NC}]{} \mathcal{B}_7'' \xrightarrow[\text{NC}]{} \mathcal{C}_3 \xrightarrow[\text{NC}]{} \mathcal{C}_3' \xrightarrow[\text{NC}]{} \mathcal{C}_3'' \rightarrow$$

$$\begin{array}{ccccccccccccc}
 & \rightarrow & \mathcal{C}_3^\bullet & \rightarrow & & & & & & & & & \mathcal{D}_2^\bullet \\
\nearrow^{\text{NC}} & & & & & \searrow^{\text{NC}} & & & & & & \nearrow^{\text{NC}} & \\
\mathcal{C}_3'' & \xrightarrow[\text{NC}]{} & \mathcal{C}_3^\circ & \xrightarrow[\text{NC}]{} & \mathcal{D}_2 & \xrightarrow[\text{NC}]{} & \mathcal{D}_2' & \xrightarrow[\text{NC}]{} & \mathcal{D}_2'' & \xrightarrow[\text{NC}]{} & \mathcal{D}_2''' & \xrightarrow[\text{NC}]{} & \mathcal{D}_2^\circ \\
 & & & & \updownarrow \text{NC} & & & & & & & & \\
 & & & & \mathcal{D}_2^\star & & & & & & & &
\end{array}$$

Though especially the tail case of $\mathcal{D}_2^\bullet$ looks very nice, symmetrical, and "easy", no progress on either case has been made till now (2008). Still, that may be because no one (including us) has tried seriously enough. On the other hand, Theorem 10 may also be used as a selection filter for ideas—if an idea should put us forward in solving Negami's conjecture, then it has to be applicable to at least one of the graphs in Π.

4 Additional remarks

Several other research papers studying planar covers of graphs, but not in a direct relation to solving Conjecture 3, have been published over the years. In [3], for instance, it is proved that no nonplanar graph has an odd-fold planar cover. In [18] it is proved that Conjecture 3 holds for all cubic graphs, but that claim is indeed a trivial corollary of Theorem 10.

In [8], a natural way of extending Conjecture 3 is outlined: the conjecture is equivalent to saying that a connected graph has a finite projective cover if and only if it is projective. Such a formulation can be easily extended to any nonorientable surface (while it is trivially false for all orientable surfaces), and little support for the Klein bottle extension was provided there [8], too. Then, the weaker projective-planar double-covering variant of this reformulation has been proved by Negami in [20], using also the idea of so called composite coverings [19].

Lastly, we return to related Fellows' notion of planar emulators [5]. Briefly, an emulator is like a cover in which the mapping of neighbours can be surjective (instead of strictly bijective). It appears very natural to extend [5] Conjecture 3 by replacing "cover" with "emulator", but very little has been done so far in it. One can immediately extend the proofs of Theorems 4 and 5 to emulators, but the same seems not so easy for the remaining results, and we know of no successful research in such a direction.

Actually, may it ever happen that a graph has an emulator but not an analogous cover? Or, is a conjecture "a graph has a finite planar emulator if and only if it has a finite planar cover" easier to solve than Negami's conjecture? We have no clue this time, but [8] shows an example of a graph which has an emulator on the triple-torus, but no cover or embedding there.

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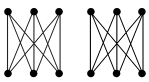
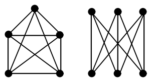
Appendix: The obstructions for projective plane

This is a list of all the 35 minor-minimal non-projective graphs [6, 1], ordered according to their significance for solving Negami's Conjeture 3.


$K_{3,3}+K_{3,3}$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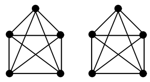
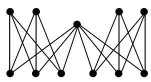

K_5+K_5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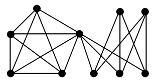
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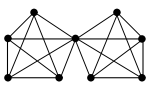
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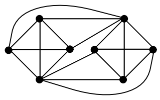
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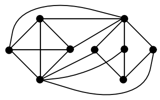
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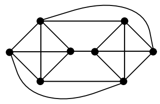
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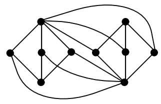
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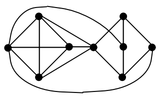
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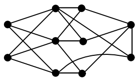
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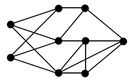
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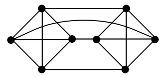
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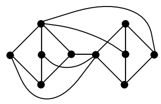
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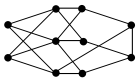
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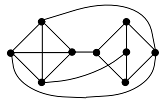
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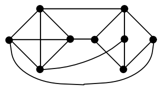
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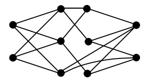
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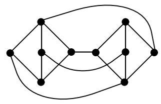
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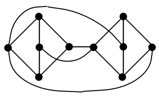
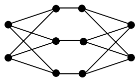
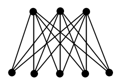
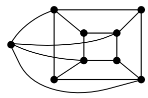
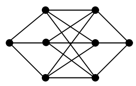
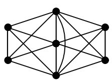

$\mathcal{F}_6$

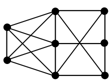

$\mathcal{G}_1$

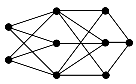

$K_{3,5}$


$K_{4,5}-4K_2$


$K_{4,4}-e$


K_7-C_4


$\mathcal{D}_3$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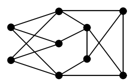
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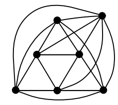
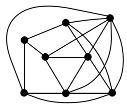

$K_{1,2,2,2}$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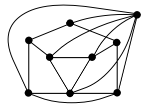
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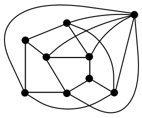
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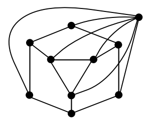
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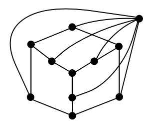
$\mathcal{E}_2$