Knotted Hamiltonian cycles in every spatial embeddings of K_8

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A *Hamiltonian cycle* in a graph is a cycle that passes through every vertex of the graph. We call a tame embedding of a graph into the 3-space R^3 a *spatial embedding* of the graph, and we call the image of a spatial embedding a *spatial graph*. A cycle of a spatial graph is said to be *knotted* if it bounds no 2-cell in R^3. Conway and Gordon [1] proved that a spatial embedding of K_7 contains exactly one knotted Hamiltonian cycle.

Joel Foisy [3] remains an open question to determine if 1 is the best lower bound for the minimum number of knotted Hamiltonian cycles in every spatial embedding of K_8. We use the result of Foisy et al. [2], we prove the complete graph K_8 contains at least 3 knotted Hamiltonian cycles in every spatial embedding.

To prove Theorem 2, we will use the following lemma.

Lemma 1. *In every spatial embedding of K_7, there exists at least 3 edges of K_7 each of which is contained in an odd number of Hamiltonian cycles with nonzero arf invariant.*

Proof. We recall the proof of Lemma 2.1 in [2], consider an arbitrary spatial embedding of K_7. Let $e_1, \ldots, e_{21}$ be the edges of K_7, and let n_i, $i = 1, \ldots, 21$ denote the number of Hamiltonian cycles with nonzero arf invariant that contain e_i. By Conway-Gordon's result [1], the given embedding contains an odd number of Hamiltonian cycles with nonzero arf invariant, let the number of such Hamiltonian cycles be $2n+1$, and then, $\sum_{i=1}^{21} n_i = 7(2n+1)$. Thus, $\sum_{i=1}^{21} n_i$ must be odd. Therefore, there must be an odd number of the n_i that is odd.

We consider for any vertex v of K_7, and let $e_{i_1}, \ldots, e_{i_6}$ be the edges incident to v. Since each Hamiltonian cycle contains exactly two edges incident to v, $\sum_{j=1}^{6} n_{i_j}$ must be even. Thus, there must be an even number of the n_{i_j} that is odd. Let $S(v)$ denote the number of n_{i_j} that is odd. Since there must be an odd number of the n_i that is odd, at least one of the n_i must be odd. Thus, there exists a vertex v_0 such that $S(v_0) \geq 2$. Hence, since there must be an odd number of the n_i that is odd, at least 3 of the n_i must be odd. □

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Theorem 2. *K_8 contains at least 3 knotted Hamiltonian cycles in every spatial embedding.*

Proof. We recall the proof of Theorem 2.2 in [2], let G_8 denote the spatial embedding of K_8, consider the spatial embedding of the subgraph induced by 7 vertices of G_8. Let v denote the eighth vertex, and let G_7 denote the subgraph on 7 vertices. By Lemma 1, there exists 3 edges e_1', e_2', e_3' of G_7 each of which is contained in an odd number of Hamiltonian cycles with nonzero arf invariant. Let w_1 and w_2 denote the vertices of e_1', and let x_1 and x_2 denote the vertices of e_2', and let y_1 and y_2 denote the vertices of e_3'. We replace e_1' with the edges (v, w_1) and (v, w_2). We denote this subdivided K_7 by G_7'. Ignoring the degree 2 vertex v, by Conway-Gordon's result [1], the embedded G_7' must have an odd number of Hamiltonian cycles with nonzero arf invariant. Since there was an odd number of Hamiltonian cycles of G_7 through the edge e_1' with nonzero arf invariant, there is an even number of Hamiltonian cycles of G_7 that do not through the edge e_1' and with nonzero arf invariant. The Hamiltonian cycles of G_7 that do not through the edge e_1' are exactly the same as the Hamiltonian cycles of G_7' that do not through the edges (v, w_1) and (v, w_2). Thus, in the embedding of G_7', there must be an odd number of Hamiltonian cycles through the edges (v, w_1) and (v, w_2) with nonzero arf invariant. These cycles are Hamiltonian cycles of G_8. Similarly, there must be an odd number of Hamiltonian cycles of G_8 that through the edges (v, x_1) and (v, x_2) with nonzero arf invariant, and there must be an odd number of Hamiltonian cycles of G_8 that through the edges (v, y_1) and (v, y_2) with nonzero arf invariant. Hence, in the G_8, there must be at least 3 knotted Hamiltonian cycles. □

References

[1] J. H. Conway and C. McA. Gordon: Knots and links in spatial graphs, J. Graph Theory. 7(1983), 446-453

[2] P.Blain, G. Bowlin, J. Hendricks, J. LaCombe, J. Foisy : Knotted Hamiltonian cycles in every spatial embeddings of complete graphs, New York Journal of Mathematics, 13(2007), 11-16.

[3] J. Foisy : Corrigendum to " Knotted Hamiltonian cycles in every spatial embeddings of complete graphs ", New York Journal of Mathematics, 14 (2008), 285-287.