Pseudo Diagrams of Knots, Links and Spatial Graphs

Ryo Hanaki*

1. Introduction

A knot, a link and a spatial graph are an embedded circle, an embedded disjoint union of some circles and an embedded graph in the 3-sphere $\mathbf{S}^3$ respectively. A *projection* P is the image of natural projection of a knot, a link and a spatial graph to the 2-sphere $\mathbf{S}^2$ such that its multiple points are only finitely many transversal double points away from the vertices. A *diagram* D is a projection P with over/under information at each double point. A diagram D uniquely represents a knot, a link or a spatial graph up to ambient isotopy. Here a double point with over/under information is called a *crossing*, in contrast a double point without over/under information is called a *pre-crossing*.

Question 1. Can we determine from P whether the original knot (link, spatial graph) is trivial or knotted?

We cannot determine it except some special cases. Because we do not know over/under information at each pre-crossing of P. For example, let P be a projection of a knot with 3 pre-crossings as illustrated in Fig. 1. Then two diagrams represent nontrivial knots and six diagrams represent the trivial knots are obtained from P.

Figure 1 Projection and diagrams obtained from it

In this paper, we study the following question.

Question 2. Which pre-crossings of P and which over/under information at them should we know in order to determine that the original knot (link, spatial graph) is trivial or knotted?

We give new definitions. A *pseudo diagram* Q is a projection P with over/under information at some pre-crossings of P. Here a pseudo diagram is possibily a pro-

*Graduate School of Education, Waseda University, 1-6-1 Nishiwaseda, Shinjuku-Ku, Tokyo 169-8050, Japan. E-mail: r.t@fuji.waseda.jp

jection or a diagram. We say that *a pseudo diagram Q' is obtained from a pseudo diagram Q* if each crossing of Q has the same over/under information as Q'. Then Q is said to be *trivial* if for any diagram D obtained from Q, D represents a trivial knot (link, spatial graph). In contrast, Q is said to be *knotted* if for any diagram D obtained from Q, D represents a nontrivial knot (link, spatial graph). For example, in Fig. 2, (a) is trivial, (b) is knotted and (c) is neither trivial nor knotted.

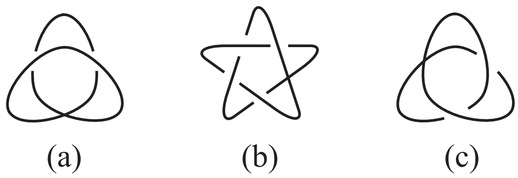

Figure 2 Pseudo diagrams

We define that the *trivializing number of P* is the minimal cardinality of C_Q where Q is a trivial pseudo diagram obtained from P and C_Q is the set of crossings of Q. Then we denote the trivializing number of P by $tr(P)$. In contrast, we define that the *knotting number of P* is the minimal cardinality of C_Q where Q is a knotted pseudo diagram obtained from P. Then we denote the knotting number of P by $kn(P)$. For example, let P be the projection in Fig. 1, then $tr(P) = 2, kn(P) = 3$.

2. A Theorem and Propositions

THEOREM 1. *Let P be a projection of a knot. Then $tr(P)$ is always even.*

PROPOSITION 2. *For any nonnegative even number n, there exists a projection of a knot with $tr(P) = n$.*

PROPOSITION 3. *There does not exist a projection of a knot whose knotting number is less than 3. For any natural number $n \geq 3$ there exists a projection of a knot with $kn(P) = n$.*

PROPOSITION 4. *For any integer z, there exists a projection P of a knot with $tr(P) - kn(P) = z$.*

In addition, we characterize a projection P_1 of a knot with $tr(P_1) = p(P_1) - 1$, a projection P_2 of a link with $tr(P_2) = 2$, a projection P_3 of a link with $tr(P_3) = p(P_3)$ and a projection P_4 of a link with $kn(P_4) = p(P_4)$ where $p(P)$ is the cardinality of the set of pre-crossings of P.

References

[1] R. Hanaki, Pseudo Diagrams of Knots, Links and Spatial Graphs, preprint.