Obstructions to shellability and related properties in dimension 2

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For a (finite) simplicial complex, *facets* are maximal faces with respect to inclusion relation, and the dimension of the complex is the dimension of the maximum facet. A simplicial complex is *pure* if all facets have the same dimension.

An ordering $F_1, F_2, \ldots, F_t$ of facets of a simplicial complex Δ is a shelling if $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ is a $(\dim F_j - 1)$-dimensional pure complex for all $2 \leq j \leq t$, and Δ is shellable if it has a shelling. (See [1]. This definition is sometimes mentioned as "nonpure shellability" distinguishing from old definition that is applicable only for the case Δ is pure.)

It is a well-known fact that a shelling induces a partition $\Delta = \dot{\bigcup}_{i=1}^{t} [R(F_i), F_i]$ by defining $R(F_i) =$ "the minimal face of F_i that is contained none of F_j with $j < i$", where $[G, F] = \{H \in \Delta : G \subseteq H \subseteq F\}$. In general, a simplicial complex Δ is called *partitionable* if it has a partition $\Delta = \dot{\bigcup}_{i=1}^{t} [R(F_i), F_i]$, where F_i is a facet and $R(F_i)$ is its face. Thus it is a well-known fact that shellability implies partitionability. (See [1].) Note that partitionability is strictly weaker than shellability.

Another property that shellability implies is sequential Cohen-Macaulayness. This property is introduced by Stanley [3] in terms of commutative algebra that generalizes Cohen-Macaulayness of pure complexes into nonpure ones, and later Duval [2] proved the definition is equivalent to the following: a simplicial complex Δ is sequentially Cohen-Macaulay if Δ_i is Cohen-Macaulay for all $0 \leq i \leq \dim \Delta$, where Δ_i is a simplicial complex consisting of all i-dimensional faces of Δ and its faces. (Note: A (pure) simplicial complex Γ is Cohen-Macaulay if $H_k(\mathrm{link}_\Gamma(\mathrm{G})) = 0$ for $k \neq \dim \mathrm{link}_\Gamma(\mathrm{G})$ for each face G of Γ, where $\mathrm{link}_\Gamma(\mathrm{G})$ is the link of G in Γ. Remark that Cohen-Macaulayness, thus also sequential Cohen-Macaulayness, depends on the characteristics of the field on which the homology group is considered.)

That shellability implies sequentially Cohen-Macaulayness, originally shown by Stanlay, is a consequence of the definition of Duval above. It is known that the converse does not hold in general.

Wachs [4] introduced the following concept: a simplicial complex Δ is an *obstruction* to shellability if Δ is nonshellable but each restriction to its proper subset of vertices is shellable. In the same way, obstructions to other properties are defined

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naturally.

There is no 0-dimensional obstruction to shellability because all 0-dimensional complexes are shellable. Further, Wachs showed the following.

(i) There is a unique 1-dimensional obstruction to shellability $2K_2$ (| |).
(ii) Obstructions to shellability exist for each dimension ≥ 1.
(iii) A 2-dimensional obstruction to shellability has at most 7 vertices. Thus there is only finite number of obstructions to shellability of dimension 2.

In this talk, we first introduce our result that determines all the 2-dimensional obstructions to shellability. Then by using this result we show the following.

Theorem 1.
(i) Obstructions to partitionability equal to obstructions to shellability for dimensions ≤ 2.
(ii) Obstructions to sequential Cohen-Macaulayness equal to obstructions to shellability for dimensions ≤ 2. (Thus, the obstructions to sequential Cohen-Macaulayness do not depend on the characteristics of the field on which the homology group is considered, for dimension ≤ 2.)

The theorem above is a corollary of the following property.

Theorem 2. *Consider a class $\mathcal{X}$ of simplicial complexes that is closed under restriction. Let $\mathcal{P}$ and $\mathcal{Q}$ be properties of simplicial complexes such that $\mathcal{P}$ implies $\mathcal{Q}$. If there exists an obstruction to $\mathcal{Q}$ in $\mathcal{X}$ which is not an obstruction to $\mathcal{P}$, then there exists an obstruction to $\mathcal{P}$ in $\mathcal{X}$ which is not an obstruction to $\mathcal{Q}$.*

By taking contraposition, if every obstruction to $\mathcal{P}$ in $\mathcal{X}$ is an obstruction to $\mathcal{Q}$, then every obstruction to $\mathcal{Q}$ in $\mathcal{X}$ is an obstruction to $\mathcal{P}$, thus we can conclude that the set of obstructions to $\mathcal{P}$ equals to set of obstructions to $\mathcal{Q}$ in the class $\mathcal{X}$. To show Theorem 1, we use Theorem 2 by letting $\mathcal{X}$ be "dimension is at most 2", $\mathcal{P}$ be shellability and $\mathcal{Q}$ be partitionability/sequential Cohen-Macaulayness, and check that each obstruction to shellability are obstruction to partitionability/sequential Cohen-Macaulayness.

References

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