Two densely embedded graphs on one surface

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Let G_1, G_2 be graphs (loops and multiple edges allowed) that embed in the surface Σ. Negami [2] introduced several different notions of how many crossings G_1 and G_2 have. In this work, we consider only common embeddings in which all vertices of both graphs are mapped to distinct points, which cannot be interior to any edge of either graph. One of Negami's notions is $\mathrm{cr}(G_1, G_2)$, which is the minimum, over all embeddings of G_1 and G_2 in Σ, of the number of crossings between the two graphs. The other of Negami's notions that concerns us is $\mathrm{cr}(\phi_1(G_1), \phi_2(G_2))$, which is the minimum, over fixed homeomorphism classes of embeddings $\phi_1(G_1)$ and $\phi_2(G_2)$ for each of G_1 and G_2, of the number of crossings between the two maps. (Two embeddings $\phi, \theta : G \to \Sigma$ are in the same homeomorphism class if there is a homeomorphism $h : \Sigma \to \Sigma$ such that $\phi = h\theta$. We will represent such a class by a particular representative.) He proved something slightly stronger than:

Theorem 1. *[2] If Σ is the orientable surface with g handles, then*

$$cr(\phi_1(G_1), \phi_2(G_2)) \leq 4g|E(G_1)||E(G_2)| \ .$$

Notice that this implies the obvious fact that if Σ is the sphere, then

$$\mathrm{cr}(G_1, G_2) = \mathrm{cr}(\phi_1(G_1), \phi_2(G_2)) = 0 \ .$$

Following up on Negami's work, Archdeacon and Bonnington [1] considered the cases of the torus and the projective plane. In both cases, they were able to get quite accurate estimates for $\mathrm{cr}(G_1, G_2)$. Essentially, they reduced the problem to computing $\mathrm{cr}_{\mathrm{sf}}(\phi_1(G_1), \phi_2(G_2))$, in which the only permitted homeomorphs $\phi'_1(G_1)$ and $\phi'_2(G_2)$ of $\phi_1(G_1)$ and $\phi_2(G_2)$ are those for which, for $\{i, j\} = \{1, 2\}$, all the vertices of G_i are in the same face of $\phi'_j(G_j)$ (hence the designation $\mathrm{cr}_{\mathrm{sf}}$). Motivated by their results, they put forward the following conjecture.

Conjecture 1. [1] For each surface Σ, there is a constant c so that

$$\mathrm{cr}(\phi_1(G_1), \phi_2(G_2)) \leq c \cdot \mathrm{cr}_{\mathrm{sf}}(\phi_1(G_1), \phi_2(G_2)) \ .$$

We invite the reader to consult [1] and [2] for further motivation and results.

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In this work we improve Negami's result as follows. The *representativity* of an embedded graph $\phi(G)$ in a surface Σ is the smallest number of intersections of a non-contractible closed curve in Σ with $\phi(G)$. Note that representativity is not defined for embeddings in the sphere.

THEOREM 2. *If Σ is an orientable surface of positive genus, then there is a constant $c' = c'(\Sigma)$, so that if, for $i = 1, 2$, $\phi_i(G_i)$ has representativity r_i, then*

$$cr(\phi_1(G_1), \phi_2(G_2)) \leq \frac{c'}{r_1 r_2}|E(G_1)||E(G_2)| \ .$$

We take the attitude that if either r_1 or r_2 is 0, then the right hand side is $+\infty$ so that the inequality holds trivially.

Negami had already observed that there is a constant $N(\Sigma, r)$ so that if $r_1 \geq r$ and $r_2 \geq r$, then

$$\mathrm{cr}(\phi_1(G_1), \phi_2(G_2)) \leq N(\Sigma, r)|E(G_1)||E(G_2)| \ ,$$

and, furthermore, if r is sufficiently large, then $N(\Sigma, r) \leq 1$.

We remark that Negami's proof depends on the deep result of Robertson and Seymour [3] that if H is a fixed embedded graph in Σ, then there is a constant $\rho(H)$ such that every graph embedded in Σ with representativity at least $\rho(H)$ contains H as a minor. Our arguments rely on more elementary facts and give explicit bounds.

We also give a counterexample in the double torus to the Archdeacon and Bonnington conjecture.

Finally, it is worth mentioning that at the heart of the proof of Theorem 2 is a result that guarantees, for a graph G embedded on an orientable surface Σ, the existence of a closed disc $D \subseteq \Sigma$, that contains all the vertices of G, and "most" of the edges.

THEOREM 3. *For each orientable surface Σ there are constants $F(\Sigma), R(\Sigma)$ with the following property. Suppose that G is embedded on Σ with representativity $r > R(\Sigma)$. Then there is a closed disc $D \subseteq \Sigma$ that satisfies the following conditions.*

(i) All the vertices of G are contained in D.
(ii) At most $F(\Sigma)|E(G)|/r$ edges of G intersect $\Sigma \setminus D$.
(iii) For each edge e, $(\Sigma \setminus D) \cap e$ is either empty or has one component.

References

[1] D. Archdeacon and C.P. Bonnington, *Two maps on one surface*, J. Graph Theory **36** (2001), no. 4, 198–216.

[2] S. Negami, *Crossing numbers of graph embedding pairs on closed surfaces*, J. Graph Theory **36** (2001), no. 1, 8–23.

[3] N. Robertson and P.D. Seymour, *Graph Minors VII. Disjoint paths on a surface.* J. Combin. Theory Ser. B **39** (1998), 212–254.