Enumerative properties of Surface Maps and Chromatic Numbers of Random Maps

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November 6, 2008

Abstract

A *map* is a graph G embedded in a surface Σ such that each component of $\Sigma - G$ is a simply connected region. Those components are called the *faces* of the map. A *circuit map*, roughly speaking, is a 2-connected planar map which is internally 3-connected. It has been shown that circuit maps share many nice properties with 3-connected planar maps. In this talk, we discuss some recent developments on the asymptotic number of surface maps which lead to the proof of a conjecture of 't Hooft in quantum physics. Those asymptotic formulas are also used to study the chromatic numbers of a random map. We will also derive an asymptotic expression for the number of circuit maps with n edges and compare it with the number of 2-connected (3-connected) planar maps.

1 Enumerative properties of surface maps

For enumeration purpose, Tutte introduced the notion of rooting a map. A map is rooted by specifying a vertex (called the *root vertex*), an edge incident with the vertex (called the *root edge*) and a side of the edge. The face on the specified side is called the *root face*. Two rooted maps are equivalent (isomorphic) if there is an automorphism of the surface which takes one map to the other and preserves the rooting.

Let M_n be the number of rooted planar maps with n edges (loops and multiple edges are allowed) and define the generating function

$$M(x) = \sum_{n \geq 0} M_n x^n.$$

In 1963 Tutte obtained the following beautiful formulas

$$M(x) = \frac{4(1 + 2\sqrt{1 - 12x})}{3(1 + \sqrt{1 - 12x})^2}, \quad \text{and } M_n = \frac{2(2n)!3^n}{n!(n+2)!}.$$

Using Stirling's formula or singularity analysis, one obtains

$$M_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n, \quad \text{as } n \to \infty.$$

In general let $M_{n,g}$ be the number of rooted maps on the orientable surface of genus g with n edges. In 1986 Bender and Canfield obtained

$$M_{n,g} \sim t_g n^{5(g-1)/2} 12^n, \ n \to \infty,$$

where t_g are positive constants satisfying complicated nonlinear recursions. We note $t_0 = 2/\sqrt{\pi}, \ t_1 = 1/24$.

There is a similar formula for the number of maps on non-orientable surfaces, with t_g replaced by another positive constant p_g.

Gao (1993) showed that many interesting families of maps satisfy asymptotic formulas of the form

$$\alpha t_g (\beta n)^{5(g-1)/2} \gamma^n,$$

where α, β, and γ are positive constants depending only on the family of maps. For example, the number $C_{n,g}$ of rooted triangulations (or cubic maps) of genus g with $3n$ edges satisfies (Gao, 1991)

$$C_{n,g} \ \sim \ 3t_g \left(6^{1/5} n\right)^{5(g-1)/2} (12\sqrt{3})^n, \ n \to \infty,$$

and the number $Q_{n,g}$, of rooted quadrangulations of genus g with $2n$ edges satisfies (Gao, 1993)

$$Q_{n,g} \ \sim \ 4^g t_g n^{5(g-1)/2} 12^n, \ n \to \infty.$$

So t_g are "universal" constants. They have been difficult to compute or estimate until very recently.

The following is from page 27 of the book *Painleve Transcendents: The Riemann-Hilbert Approach*, by Fokas, Its, Kapaeve, and Novokshenov (AMS 2006):

It was shown in the seminal paper of D. Bessis, C. Itzykson, and J. B. Zuber [**BIZ**], that the integral (60) (which is well defined for all $t \geq 0$) admits the following formal expansion[4] over N^{-2},

(62) $$\log \frac{Z_N(t)}{Z_N(0)} \sim \sum_{g=0}^{\infty} N^{2-2g} E_g(t),$$

where

(63) $$E_g(t) = \sum_{n \geq 1} (-t)^n \frac{\kappa_g(n)}{n!}.$$

The coefficients κ_n of the series (62) have a profound topological meaning, namely

(64) $$\kappa_g(n) = \#\{\text{connected maps of genus } g \text{ with } n \text{ 4-valent vertices}\}.$$

On page 29 of the same book:

(73) $$E_g(t) \sim e_g \left(t + \frac{1}{48}\right)^{5/2(1-g)} + \text{constant}.$$

Indeed, if we assume (73) and the validity of (62) for $-1/48 < t < 0$ (the absence of the Stokes phenomenon for the partition function $Z_N(t)$!) then by a direct formal calculations we find (cf. [**FGZ**])

$$\log \frac{Z_N(t)}{Z_N(0)} - N^2 \text{ const} \sim \sum_{g=0}^{\infty} N^{2-2g} e_g \left(t + \frac{1}{48}\right)^{5/2(1-g)}$$

$$= \sum_{g=0}^{\infty} N^{2-2g} e_g \left(-c_0 x N^{-4/5}\right)^{5/2(1-g)} = \sum_{g=0}^{\infty} \hat{c}_g (-x)^{5/2-5g/2}.$$

The second derivative of the last series with respect to x is of the form of the series (68). This, in view of (66), can be considered as a formal confirmation of the estimate (73) (see the survey [**FGZ**] and references therein for more details). In turn, the estimate (73), following a standard reasoning, implies the following estimate for the combinatorial numbers (64):

(74) $$\kappa_g(n) \sim Cn! \frac{48^n}{n^{1+5/2(1-g)}}.$$

The above arguments are, of course, formal. The status of estimate (73), and hence of (74), remains that of a conjecture. Nevertheless, the current level of development of the Riemann-Hilbert techniques, and the experience with other combinatorial problems e.g. in random permutations [**BDJ**], suggest that all the gaps in the above construction will be soon filled. It should be mentioned, that the estimate (73) is also predicted by the continuum string theory (Liouville gravity, see [**FGZ**]).

We note that estimate (74) follows immediately from the above formula for $Q_{n,g}$ by noting

$$\kappa_g(n) = \frac{n! 2^{2n-1} Q_{n,g}}{4n} \sim \frac{4^g t_g}{8} n! \frac{48^n}{n^{1+(5/2)(1-g)}}.$$

Also, (73) is not quite right. The correct form should be

$$E_g(t) \sim e_g \left(t + \frac{1}{48}\right)^{(5/2)(1-g)} + f_g \left(t + \frac{1}{48}\right)^{(5/2)(1-g)+1/2}, \quad g > 1,$$

and

$$E_1(t) \sim e_1 \ln \left(t + \frac{1}{48}\right) + f_1.$$

Using algebraic combinatorics and KP-hierarchy, Goulden and Jackson (08) derived the following remarkably simple recursion: for $(n,g) \neq (-1,0)$,

$$H_{n,g} \;=\; \frac{4(3n+2)}{n+1}\Big(n(3n-2)H_{n-2,g-1} + \sum_{i=-1}^{n-1}\sum_{h=0}^{g} H_{i,h}H_{n-2-i,g-h}\Big),$$

where $H_{n,g} = (3n+2)C_{n,g}$ and $C_{n,g}$ is the number of rooted cubic maps (or triangulations) with $3n$ edges on an orientable surface of genus g.

Using the Goulden-Jackson recursion and the asymptotic formula for $C_{n,g}$, Bender, Gao, and Richmond (08) obtained

$$t_g \sim \frac{40K\sin(\pi/5)}{\sqrt{2\pi}}\left(\frac{1440g}{e}\right)^{-g/2}, \;\; g \to \infty.$$

where $K \doteq 0.10$ is a constant.

Using the above estimate for t_g, Garoufalidis, Le and Marino (08) proved a conjecture of 't Hooft (82) about analyticity of free energy. They also noticed that there is a connection between t_g and Painleve I ODE, which leads to

$$t_g \sim \frac{1}{\pi}\sqrt{\frac{30}{\pi}}\left(\frac{1440g}{e}\right)^{-g/2}, \;\; g \to \infty.$$

2 Locally planar maps and chromatic number of random maps

We first recall the definition of locally planar maps and some related results.

- The face width (representativity) of a map is the minimum number of intersections of a noncontractible cycle in the surface with the graph.
- A map in a given surface is called *locally planar* if it has sufficiently large face width, that is, its face width exceeds some constant only depending on the surface.
- Bender-Gao-Richmond (94): On any fixed surface, a random map (from many interesting families) is asymptotically almost surely (a.a.s.) locally planar. In fact the face width of a random map with n edges is a.a.s. of the order $\ln n$.
- Richmond and Wormald (95) proved that a random map a.a.s. has no symmetry, and hence a property holds a.a.s. for a random rooted map in a given family if and only if it holds a.a.s. for a random (unrooted) map.
- Robertson-Vitray (90): If M is locally planar then it is a minimum genus embedding of its underlying graph $G(M)$, and if $G(M)$ is also 3-connected then M is the only embedding of $G(M)$ on the same surface. Hence for a family of 3-connected maps, an a.a.s result for random maps carries to the underlying graphs.
- Mohar-Robertson (01) On any fixed surface, the number of 3-representative embeddings of a 3-c graph is bounded by a constant (depending on the surface).
- Thomassen (92): every locally planar map is 5-colorable (vertex coloring or face coloring by duality). This implies that a random 3-c graph of bounded genus a.a.s. has a 5-flow.
- Hutchinson (95): every locally planar quadrangulation on an orientable surface is 3-colorable. Again this implies that a random 3-c 4-regular graph of bounded genus a.a.s. has a 3-flow.

Question 1 Let $p_j(\Sigma)$ be the limiting probability (if exists) that a random map on a fixed surface Σ has chromatic number j. We note $p_4(\Sigma) + p_5(\Sigma) = 1$. What are $p_4(\Sigma)$ and $p_5(\Sigma)$? Bender-Gao-Richmond (1994) actually conjectured that $p_4(\Sigma) = 1$.

Question 2. It follows from Hutchinson's result that $p_2(\Sigma) + p_3(\Sigma) = 1$ for a random quadrangulation of an orientable surface Σ.

Fisk and Mohar (94): all locally planar quadrangulations of a given non-orientable surface are 4-colorable.

Hence $p_2(\Sigma) + p_3(\Sigma) + p_4(\Sigma) = 1$ for any non-orientable surface Σ.

What are the values of those limiting probabilities (if exist)?

Some of the above questions can be answered by noting

- (Bender-Canfield, 86) bipartite quadrangulations with $2n$ edges:

$$|\mathcal{F}_n| \sim t_g n^{5(g-1)/2} 12^n$$

- (Gao, 93) quadrangulations with $2n$ edges:

$$|\mathcal{F}_n| \sim 4^g t_g n^{5(g-1)/2} 12^n.$$

It follows from the above asymptotic results that, for random quadrangulations on the orientable surface of genus g, $p_2 = (1/4)^g$, $p_3 = 1 - (1/4)^g$.

- (Nakamoto-Negami-Ota,04): there are 4-chromatic locally planar qudrangulations of each non-orientable surface. Archdeacon-Hutchinson-Nakamoto-Negami-Ota (01) characterized quadrangulations of the torus and Klein bottle with chromatic number 3. Youngs (96) proved that a quadrangulation of the projective plane has chromatic number equal to either 2 or 4. Hence for random quadrangulations of the projective plane, we have

$$p_2 = 1/2, \; p_3 = 0, \; \text{and } p_4 = 1/2.$$

- For random quadrangulations of the non-orientable surface of Euler genus g, we have $p_2 = (1/4)^g$, $p_3 + p_4 = 1 - (1/4)^g$.

3 Enumeration of circuit maps

In 1966 Barnette [2] introduced a set of graphs, called *circuit graphs*, which are obtained from 3-connected planar graphs by deleting a vertex. Circuit graphs have nice closure properties which make them easier to deal with than 3-connected planar graphs for studying some graph-theoretic properties. Circuit graphs and 3-connected planar graphs share many interesting properties which are not satisfied by general 2-connected planar graphs. For example, Barnette [2] proved that every circuit graph has a spanning tree with maximum degree at most 3 (called a *3-tree*); This is strengthened by Gao and Richter [14] who showed that every circuit graph contains a closed walk visiting each vertex once or twice (called a *2-walk*); Jackson and Wormald [17] showed that the existence of a 2-walk in a graph implies the existence of a 3-tree; It is also known that circuit graphs contain long cycles [18, 9, 15] and they contain 2-connected spanning trees with small maximum degree [3, 13]. Very recently Nakamoto et al [24] showed that every circuit graph contains a 3-tree with few vertices of degree 3.

For enumeration purpose we shall use the following equivalent definition of circuit graphs (Gao-Richter, 94) : A *circuit graph* is an ordered pair (G, C) such that

(1) G is a 2-connected graph and C is a cycle in G;

(2) there is an embedding of G in the plane such that C bounds a face;

(3) if (H, K) is a 2-separation of G, then $C \nsubseteq H$ and $C \nsubseteq K$.

A *simple circuit map* is a circuit graph (G, C) embedded in the plane so that C bounds the exterior face. It is rooted by choosing the exterior face as the root face. It is also convenient to consider *circuit maps* which are defined almost identically to simple circuit maps except that there could be digons adjacent to the root face.

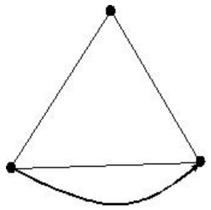

Figure 1: Circuit maps with multiple edges

We shall use the following well-known bijection ϕ between rooted maps and rooted quadrangulations. Insert a vertex inside each face of a map M and join it to each vertex on the face to subdivide the face into triangles. Removing the edges of M gives the quadrangulation $\phi(M)$. The rooting of $\phi(M)$ can be chosen in the following canonical way: the vertex f_0 inside the root face of M is the root vertex of $\phi(M)$, the edge joining f_0 and the root vertex of M is the root edge of $\phi(M)$.

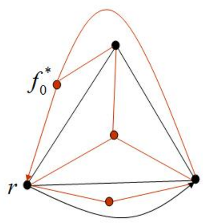

Figure 2: Bijection between rooted maps and rooted quadrangulations

It is easy to check that ϕ has the following properties.

- M is 2-connected if and only if $\phi(M)$ has no multiple edges,
- M is 3-connected if and only if $\phi(M)$ has no multiple edges and no separating quadrangles.
- M is a circuit map if and only if $\phi(M)$ has no multiple edges and no root-separating quadrangles.

Let $Q_{i,j,k}$ be the number of rooted quadrangulations with no multiple edges, and with i red vertices, j blue vertices and root vertex degree k. Define the generating function

$$Q(x,y,z) = \sum_{i,j,k\geq 2} Q_{i,j,k} x^{i-1} y^{j-1} z^k.$$

Similarly define $\bar{Q}_{i,j,k}$ and $\bar{Q}(x,y,z)$ for rooted root-simple quadrangulations. By Lemma 4 and [8, (3.8)], we have

$$\begin{aligned} & Q^2(x,y,z) + ((1-z)(1-xz) + yz - zQ(x,y,1))\, Q(x,y,z) \\ = \;& yz^2(x(1-z) + Q(x,y,1)). \end{aligned} \tag{1}$$

Also from [8], we have

$$Q(x,y,1) = uv(1-u-v), \tag{2}$$

where u and v are unique power series in x and y defined by

$$x = u(1-v)^2, \quad y = v(1-u)^2. \tag{3}$$

We note that $Q(x,y,x)$ are now determined by (1)

Let $\bar{C}_{i,j,k}$ be the number of rooted simple circuit maps with i vertices, j faces and root face degree k, and let

$$\bar{C}(X,Y,Z) = \sum C_{i,j,k} X^{i-1} Y^{j-1} Z^k.$$

Define $C(X,Y,Z)$ analogously for rooted circuit maps with at least 3 vertices. We have

Theorem 1 *Let*

$$X = Q(x,y,1)/y, \;\; Y = Q(x,y,1)/x, \;\; Z = xyz/Q(x,y,1).$$

Then

$$C(X,Y,Z) = Q(x,y,z), \tag{4}$$

$$\bar{C}(X,Y,Z(1+Y)) = C(X,Y,Z) - XY(1+Y)Z^2. \tag{5}$$

Proof: For any rooted quadrangulation Q, call a root-separating quadrangle *maximal* if it is not inside another root-separating quadrangle. It is easy to see that the interiors of maximal root-separating quadrangles are pairwise disjoint. Therefore, removing all vertices and edges in the interior of each maximal root-separating quadrangle yields a root-simple quadrangulation, and this process can be reversed by replacing each face of a root-simple quadrangulation, that is not incident with the root vertex, with an arbitrary quadrangulation.

Hence

$$\begin{aligned} Q(x,y,z) &= \sum \bar{Q}_{i,j,k} x^{i-1} y^{j-1} z^k \,(Q(x,y,1)/xy)^{i+j-2-k} \\ &= \bar{Q}(Q(x,y,1)/y, Q(x,y,1)/x, xyz/Q(x,y,1)). \end{aligned}$$

Now (4) follows from Lemma 4. We Note that there are exactly two circuit maps with two vertices, the one with two parallel edges and the other one with three parallel edges. Also circuit maps with more than two vertices can be generated from simple circuit graphs with more than two vertices by replacing some edges on the root face with a digon. This gives (5). ∎

Let $\bar{c}_{n,k}$ and $c_{n,k}$ be the number of rooted circuit maps and rooted simple circuit maps, respectively, that have n edges and root face degree k. It follows from Euler's formula for planar maps that

$$C(X, X, Z) = \sum c_{n,k} X^n Z^k, \tag{6}$$
$$\bar{C}(X, X, Z) = \sum \bar{c}_{n,k} X^n Z^k. \tag{7}$$

Setting $u = v$ in (2) and (3), we obtain $x = y$ and

$$x = u(1-u)^2,\ Q(x, x, 1) = u^2(1-2u). \tag{8}$$

Setting $u = v$ and $Z = 1$, we obtain $X = Y = \frac{u(1-2u)}{(1-u)^2}$, $z = (1-2u)(1-u)^{-4}$, and

$$\begin{aligned} C(X, X, 1) &= \frac{u(3u^2-3u+1)(u^3-u^2-2u+1)}{2(1-u)^6} \\ &\quad -\frac{u(u^2-3u+1)}{2(1-u)^6}\sqrt{(u^3-u^2-2u+1)(1-6u+11u^2-7u^3)} \\ &= X^2 + 2X^3 + 4X^4 + 10X^5 + 27X^6 + 79X^7 + 243X^8 + \cdots. \end{aligned}$$

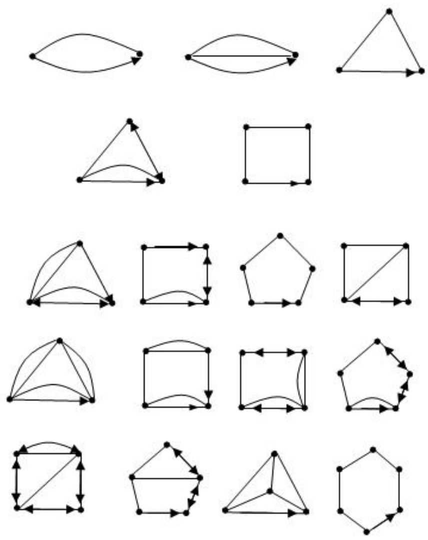

Figure 3: A list of small rooted circuit maps

Let $x_0 > 0$ be the singularity of $C(X, X, 1)$ closest to the origin. We find that x_0 is determined by the equations

$$X = \frac{u(1-2u)}{(1-u)^2},\ 1 - 6u + 11u^2 - 7u^3 = 0.$$

We obtain

$$x_0 = -(1/6)(100 + 12\sqrt{69})^{1/3} - 2/(3(100 + 12\sqrt{69})^{1/3}) + 4/3 \doteq 0.24512233. \tag{9}$$

Near $X = x_0$, we have

$$C(X, X, 1) = -a_1(1 - X/x_0)^{1/2} + a_2(1 - X/x_0) + O\left((1 - X/x_0)^{3/2}\right),$$

where $a_1 \doteq 0.147041065$.

Hence

$$C_n \sim \frac{a_1}{2\sqrt{\pi}} n^{-3/2} x_0^{-n}.$$

We shall prove the following

Theorem 2 *(i) The number of rooted simple circuit maps is asymptotic to*

$$\frac{6}{25\sqrt{\pi}} n^{-5/2} 4^n.$$

(ii) The number of rooted circuit maps is asymptotic to

$$\frac{0.147}{2\sqrt{\pi}} n^{-3/2} 4.08^n.$$

To obtain $\bar{C}(X, X, 1)$, we set $u = v$ and $Z = 1/(1 + Y)$. Then

$$\bar{C}(X, X, 1) = Q(x, x, z) - \frac{X^2}{1 + X},$$

where

$$z = \frac{1 - 2u}{(1 - u)^4}.$$

Thus

$$Q(x, x, z) = \frac{u^2(1 - 2u - u^2)}{(1 - u - u^2)^2},$$

and

$$\bar{C}(X, X, 1) = \frac{u^2(1 - 2u - u^2)}{(1 - u - u^2)^2} - \frac{X^2}{1 + X},$$

where

$$u = \frac{1 - \sqrt{1 - 4X}}{3 - \sqrt{1 - 4X}}.$$

Using Maple, we obtain the following expansion

$$\bar{C}(X, X, 1) = X^3 + X^4 + 3X^5 + 7X^6 + 19X^7 + 54X^8 + \cdots$$

It follows that the dominant singularity of $C(X, X, 1)$ is $X = 1/4$ at which $C(X, X, 1)$ has the following asymptotic expansion

$$C(X, X, 1) + \frac{X^2}{1 + X} = 2/25 - (36/125)(1 - 4X) + (8/25)(1 - 4X)^{3/2} + O\left((1 - 4X)^2\right).$$

Hence

$$c_n \sim \frac{6}{25\sqrt{\pi}} n^{-5/2} 4^n.$$

It is also interesting to compare c_n with the number of rooted 2-connected simple maps. The following result may not be new; however, we are unable to find it in the literature, so we include its proof here for self completeness.

Lemma 1 *The number of rooted 2-connected simple maps is asymptotic to*

$$\frac{352}{675}\sqrt{\frac{1}{15\pi}}n^{-5/2}(729/128)^n.$$

Proof: Let $B(x)$ be the generating function for rooted 2-connected maps, and $\bar{B}(x)$ for rooted 2-connected simple maps. Then

$$B(x) = Q(x,x,1) + x = u(1-u-u^2),\ x = u(1-u)^2.$$

Since each rooted 2-connected map can be obtained from a rooted 2-connected simple map by replacing some edges with a rooted 2-connected map, we have $B(x) = \bar{B}(x(1+B(x)))$. This gives the following parametric expression for $\bar{B}(X)$:

$$\bar{B}(X) = u(1-u-u^2),\ X = u(1-u)^3(1+u)^2.$$

Hence u and $\bar{B}$ are both algebraic functions of X. The dominant singularity of $u(X)$ and $\bar{B}(X)$ is obtained by solving

$$X = u(1-u)^3(1+u)^2,\ X'(u) = 0,$$

which gives $u = 1/3$ and $X = 128/729$. Also $\bar{B}(X)$ has the following asymptotic expansion at $X = 128/729$:

$$\bar{B}(X) = \frac{5}{27} - \frac{32}{135}\left(1 - \frac{729}{128}X\right) + \frac{1408\sqrt{15}}{30375}\left(1 - \frac{729}{128}X\right)^{3/2} + \cdots.$$

Now the lemma follows immediately from Darboux's theorem. ∎

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