Rigidity and separation indices of embedded graphs

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Introduction

Let $\mathcal{L}_n$ be the lattice of partitions of the n-set $[n] = \{1, 2, \ldots, n\}$, with minimal element 0 (partition into singletons), and maximal element 1 (having one block only).

Let G be a graph of order n with vertex set $V(G) = [n]$, and let $\Gamma = \mathrm{Aut}(G)$ be the group of automorphisms of G with its natural action on $V(G)$. For $v \in V(G)$ let $\Gamma_v \leq \Gamma$ be the *stabilizer* of v in Γ, and let P_v be the corresponding partition of $V(G)$ into the orbits of Γ_v.

The *separation index* of G, denoted by $\mathrm{sep}(G)$ is the minimum k such that there exists a vertex set $U \subseteq V(G)$ of cardinality k so that

$$\bigwedge_{u \in U} P_u = 0 \tag{1}$$

in $\mathcal{L}_n$. We also say that the vertices of U (as in (1)) *separate* G.

On the other hand we define the *rigidity index* of G, denoted by $\mathrm{rig}(G)$, as the minimum k, such that there exists a vertex set $W \subseteq V(G)$ of cardinality k so that

$$\bigcap_{w \in W} \Gamma_w = \{\mathrm{id}\}. \tag{2}$$

The vertices of W are said to *fix* the graph G.

Let us first state some examples: $\mathrm{sep}(K_n) = \mathrm{rig}(K_n) = n - 1$, $\mathrm{sep}(K_{m,n}) = \mathrm{rig}(K_{m,n}) = m + n - 2$ (for $n, m \geq 2$), $\mathrm{sep}(\overline{G}) = \mathrm{sep}(G)$, and also $\mathrm{rig}(\overline{G}) = \mathrm{rig}(G)$. As a subset separating vertices of G also fixes G we have $\mathrm{rig}(G) \leq \mathrm{sep}(G)$. Strict inequality may occur, and Paley graphs may serve as the extreme cases [1] having rigidity index equal to 2 and arbitrarily large separation indices.

Every 3-connected planar graph is uniquely embeddable in the plane and identity is the only authomorphism which fixes three consecutive vertices of any facial walk. This implies that rigidity index of a 3-connected planar graphs is at most 3. Vince [4] proved that also separation index of a 3-connected planar graph is bounded by 3.

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Results

We looked for classes of graphs with bounded separation or rigidity index. We were able to show that these indices cannot be too large compared to the order of the authomorphism group. Namely, there exists an integer function $f : \mathbb{N} \to \mathbb{N}$ so that for every graph G we have $\mathrm{rig}(G) \le \mathrm{sep}(G) \le f(|\mathrm{Aut}(G)|)$.

In order to control the order of the authomorphism group of a graph one can consider graphs which nicely embed in a fixed surface Σ. We say that G is *polyhedrally embeddable* in Σ, $G \in P_\Sigma$, if G is 3-connected and admits an embedding in Σ with face-width at least 3. In this case the intersection of every pair of faces (of such an embedding of G in Σ) is either a vertex, a single edge, or empty.

Mohar and Robertson [3] have shown that a graph can only have a bounded number of different polyhedral embeddings in Σ, if Σ is fixed. Together with Hurwitz' theorem [2] this bounds the order of the automorphism group, thus:

Theorem 1. *For every surface Σ with negative Euler characteristic there exists a constant s_Σ so that the separation index of every $G \in P_\Sigma$ satisfies* $\mathrm{sep}(G) \le s_\Sigma$.

There exist 3-connected projective-planar graphs as well as 4-connected toroidal graphs with arbitrarily large rigidity index. Raising connectivity to 5 does help — an embedding of a 5-connected graphs in a surface Σ is sufficiently close to a polyhedral one. We were able to prove:

Theorem 2. *For every surface Σ there exists a constant p_Σ so that the rigidity index of every 5-connected graph embeddable in Σ and also every graph $G \in P_\Sigma$ satisfies* $\mathrm{rig}(G) \le p_\Sigma$.

Let us mention at the end that Theorems 1 and 2 do not extend to more general minor closed families of graphs, say graphs of bounded tree-width. Consider the strong product $K_k \boxtimes T$ of a complete graph K_k and a tree T: $K_k \boxtimes T$ is a k-connected graph, its tree width is at most $2k-1$, yet its rigidity (and also separation) index is at least $(k-1)|V(T)|$.

References

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