The Link Component Number of Suspended Trees

Toshiki Endo *

Graphs are often used as a research tool in knot theory. This is because there is a one-to-one correspondence between a link diagram and an edge-signed plane graph, as described bellow. See [1] for details.

Let L be a link diagram in a sphere $\mathbf{S}^2$. The projection of L is a 4-regular plane graph, and sometimes called a *universe* of L. We color the faces black and white. From this coloring, we get an edge-signed plane graph G_L, where its vertices are the black faces and two vertices are joined by a signed edge if they share a crossing. Each edge is given a plus or minus sign as shown in Fig. 1. Conversely, for a given edge-signed plane graph G, we can associate a link diagram L such that $G_L = G$ by considering the medial graph of G.

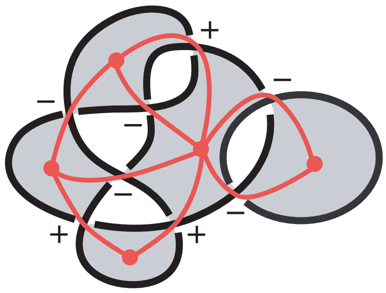

Figure 1 A link diagram L and its graph G_L

Our research interest is to explore when a graph represent a knot and when a link. For this purpose, we may only consider unsigned graphs, since ignoring signs of edges does not change the number of link components. We prepare one term: for a plane graph G, we call the number of components of the associated link a *link component number* of G, and denote by $l(G)$. Our research goal is to find a method to determine the link component number $l(G)$ for a given plane graph G.

Although it is not a desirable one, there is a solution of the problem above. Let $T(G, x, y)$ be the Tutte polynomial of a plane graph G. It is known that the equation $T(G, -1, -1) = (-1)^{|E(G)|}(-2)^{l(G)-1}$ holds [6]. Our aim is to find out a relation between geometric structures and the link component number of a graph. There are several early studies along this line. See [2] [3] [5] for examples.

In this talk, we consider a certain graph family generated by a tree, and completely determine its link component number. For a tree T, a vertex of T is called

*Research Institute for Mathematics and Information Science, Jiyu Gakuen College, 1-8-15 Gakuencho, Higashikurume-Shi, Tokyo 203-8521, Japan. E-mail: `end@prf.jiyu.ac.jp`

a *leaf* if it is of degree one. We define a *suspended tree* S_T to be a plane graph generated by the following process. Start with a tree T in $\mathbf{S}^2$. Add a new vertex v. From v, add one edge to each leaf of T. An example is given in Fig. 2. The tree T is drawn with bold lines.

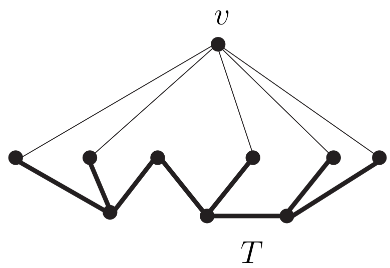

Figure 2 A suspended tree

Our theorem states that the link component number of S_T does not depend on embeddings of T into $\mathbf{S}^2$, and depend on essentially the number of $K_{1,2}$'s of T. Details will be explained in our talk.

Finally we shall briefly describe a relationship between suspended trees and knot theory. A link L is called *arborescent* (or *algebraic*) if L is formed by taking the numerator closure of an tangle obtained by additions and multiplications of rational tangles. See [1] for details. It is known that a link L is arborescent if and only if the associated graph G_L has a vertex v such that $G_L - v$ is a tree [4]. Thus, as an application of our theorem, we may determine the link component number of an arborescent link via its graph.

References

[1] C. C. Adams, The knot book, W. H. Freeman and Company, New York, 1994.

[2] S. Eliahou, F. Harary and L. H. Kauffman, Lune-free knot graphs, *J. Knot Theory Ramifications* **17** (2008), 55–74.

[3] E. G. Mphako, The component number of links from graphs, *Proc. of Edin. Math. Soc.* **45** (2002), 723–730.

[4] K. Murasugi and J. H. Przytycki, An index of a graph with applications to knot theory, *Mem. Amer. Math. Soc.* **106** (1993), no. 508, x+101 pp.

[5] S. D. Noble and D. J. A. Welsh, Knot graphs, *J. Graph Theory* **34** (2000), 100–111.

[6] W. Schwarzler and D. J. A. Welsh, Knots, matroids and the ising model", *Math. Proc. Camb. Phil. Soc.* **113** (1993), 107–139