Transition graphs and some applications

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Abstract

Transition graphs are an algebraic way of constructing embeddings of graphs in surfaces. They are equivalent to embedded voltage graphs but are more convenient in certain situations. We describe how transition graphs can be used to construct embeddings and discuss some of their advantages. As applications, we discuss how transition graphs can be used to construct embeddings of complete bipartite graphs with very specific properties. From these we can construct minimum genus embeddings of certain complete tripartite graphs and other related graphs, such as joins of a complete graph with an empty graph.

1 Introduction

In many parts of mathematics, the choice of representation of a mathematical structure can make it easier or harder to use. A classic example in topological graph theory is the use of current graphs to construct minimum genus embeddings of complete graphs in the proof of the Heawood Map Coloring Theorem [5]. Current graphs are not the most direct algebraic construction for describing a graph embedding. Embedded voltage graphs are of equal power (being related to current graphs by duality) and are more straightforward. However, current graphs were invented and used earlier than voltage graphs precisely because they were convenient for constructing triangular embeddings of complete

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graphs. Voltage graphs would have been much harder to manipulate for this particular purpose.

Over the past few years we have been working on determining the minimum genus of complete tripartite graphs and related graphs, such as joins of a complete graph with an empty graph. In this situation, neither voltage graphs nor current graphs are convenient. Instead, we use a third type of algebraic construction for graph embeddings, which we call *transition graphs*. Our goal here is to explain what transition graphs are, and why they are convenient for constructing certain types of graph embedding.

2 What is a transition graph?

Transition graphs are an algebraic construction of equal power to embedded voltage graphs or their duals, current graphs (voltage and current graphs are described in [4]). In fact a transition graph can be regarded as just the medial graph of an embedded voltage graph. Archdeacon [1] gave an algebraic construction for graph embeddings using medial graphs, but his construction applies voltages to the edges of the medial graph, whereas ours applies voltages to the vertices of the medial graph, so the constructions are different.

A formal definition of general transition graphs can be found in [3]. Here we proceed a less formally, using an example, shown in Figure 1. For a transition graph we need a group Γ, a directed graph D in which every vertex has indegree and outdegree 2, and a partition of the edges of D into directed closed walks. For each vertex v of D one of the two directed walks passing through v should be designated as the *reference at* v, v should be designated as either untwisted (open) or twisted (solid), and v should be labelled with an element of Γ, called its *voltage*. From this we will obtain a *derived graph embedding*.

In Figure 1 there are only two directed closed walks, A (solid) and B (dashed), which are actually directed cycles, and every vertex is incident with both A and B. Thus, we may take A to be the reference at every vertex. Every element of Γ is used exactly once as the label of a vertex. None of these properties is necessary for general transition graphs. But they imply that the derived graph is complete bipartite, as we shall see. Moreover, the group ($\mathbb{Z}_8$) is abelian and even cyclic. If all of the properties in this paragraph are satisfied we say we have a *cyclic transition graph*.

Although cyclic transition graphs are a very special case, they allow us to construct embeddings of $K_{m,m}$ which can be extended to embeddings of related graphs such as $\overline{K_m} + K_m$, or $K_{m,m,n}$ with $n \leq m$. Using other techniques, the embeddings of $K_{m,m,n}$ can be further extended to embeddings of general complete tripartite graphs $K_{l,m,n}$ with $l \geq m \geq n$.

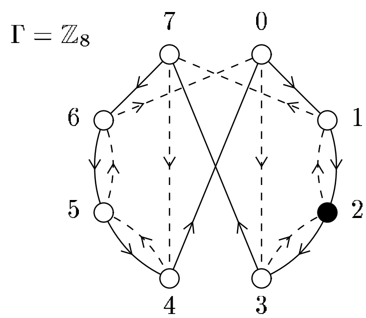

Figure 1

The derived graph embedding is obtained from the transition graph as follows.

Vertices. Each directed closed walk yields $|\Gamma|$ vertices in the derived graph, indexed by the elements of Γ. So in the example, the derived graph has sixteen vertices: a_i, $i \in \mathbb{Z}_8$, from A, and b_i, $i \in \mathbb{Z}_8$, from B.

Edges. Each vertex v of the transition graph generates $|\Gamma|$ edges in the derived graph, going between the two classes of vertices corresponding to the directed closed walks through v. The exact edges depend on the voltage of v. In the example, a vertex of label k represents 8 edges from a_i (since A is the reference) to b_{i+k} for $i \in \mathbb{Z}_8$. For example, the vertex labelled 2 represents the family of edges a_0b_2, a_1b_3, …, a_7b_1 which have *slope* equal to 2.

Since each $k \in Z_8$ occurs exactly once as a vertex label, we get each edge of the form a_ib_{i+k} exactly once in the derived graph, showing that the derived graph of the example is $K_{8,8}$.

Topological information. The derived embedding is defined to be cellular: each face is an open 2-cell. The directed closed walks indicate the rotation order of the edges around each class of vertex. In the example, the solid directed cycle A is (01237654), so around each vertex a_i the rotation of edges is in the order a_ib_i, a_ib_{i+1}, a_ib_{i+2}, a_ib_{i+3}, a_ib_{i+7}, a_ib_{i+6}, a_ib_{i+5}, a_ib_{i+4}. Similarly the dashed cycle B is (03217456), so around each vertex b_i the rotation of edges is in the order a_ib_i, $a_{i-3}b_i$, $a_{i-2}b_i$, $a_{i-1}b_i$, $a_{i-7}b_i$, $a_{i-4}b_i$, $a_{i-5}b_i$, $a_{i-6}b_i$: the minus sign in each $a_{i-k}b_i$ is needed because A, not B, is the reference at each vertex labelled k.

Moreover, solid vertices correspond to edges in the derived graph that are ‘twisted’ or orientation reversing. Since the vertex labelled 2 is solid, the edges of the form a_ib_{i+2} are twisted in the embedding, meaning that we must reverse the direction in which we follow the rotation around a vertex when we traverse one of these edges.

The above completely describes the derived embedding. A procedure for tracing the faces can be given, but for the examples we present there will be a simpler way of determining the faces of the resulting embedding, as we describe in Section 4 below.

3 Advantages of transition graphs

Transition graphs have the following advantages.

- Transition graphs can be built up from small patterns representing particular groups of faces.
- Transition graphs can be used to build whole families of embeddings at once, by making substitutions involving small patterns.
- Transition graphs can be used to build relative (partial) embeddings, which can be completed with non-algebraic constructions, in situations where a completely algebraic construction is impossible.
- Transition graphs allow very fine tuning of the properties of an embedding, which may then allow further manipulations to be performed.

We illustrate these in the following sections. All of our examples will involve cyclic transition graphs. Thus, the group is henceforth always assumed to be $\mathbb{Z}_m$ where m is the number of vertices in the transition graph. We are always constructing embeddings of $K_{m,m}$, and we often modify these embeddings to obtain embeddings of related graphs such as $K_{m,m,n}$.

4 Using small patterns

Transition graphs can be built up from small 'patterns' which represent particular types of faces in the derived embedding. In particular, we commonly use the patterns shown in Figure 2 in cyclic transition graphs with group $\mathbb{Z}_m$ and derived graph $K_{m,m}$.

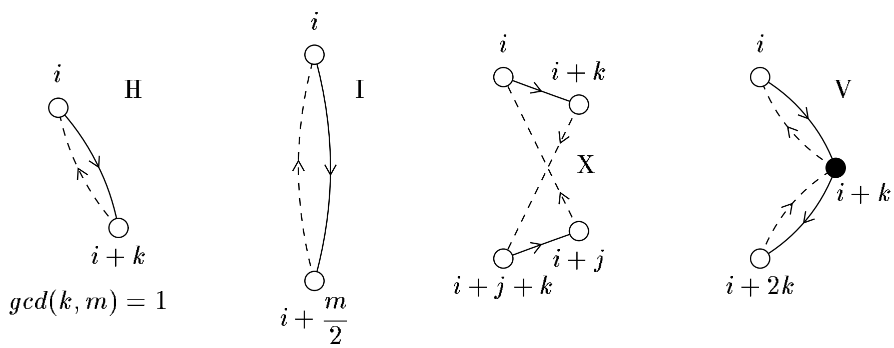

Figure 2

Each type H pattern produces one hamilton cycle face in $K_{m,m}$, of the form $(a_0 b_{i+k} a_k b_{i+2k} a_{2k} \dots b_i)$. Type I patterns occur only when m is even, and produce $m/2$ faces of degree 4 of the form $(a_j b_{i+j+m/2} a_{j+m/2} b_{i+j})$. Type X patterns produce m faces of degree 4, as do type V patterns. Type V patterns require a solid (twisted) vertex and always produce a nonorientable embedding. The other patterns may be used in constructing either orientable or nonorientable embeddings. For simplicity, we frequently use patterns of type H, X or V for which $k = 1$.

Our original example, Figure 1, can be thought of as built up of two type H patterns (with vertices $4, 5$ and $5, 6$), two type X patterns (one with vertices $0, 1, 6, 7$ and the other with vertices $0, 3, 4, 7$), and one type V pattern (with vertices $1, 2, 3$). Therefore, it corresponds to an embedding of $K_{8,8}$ with two hamilton cycle faces (from the type H patterns) and 24 faces that are 4-cycles (from the type X and type V patterns). Since a type V pattern is used, the embedding is nonorientable. By inserting a vertex joined to all original vertices in each of the two hamilton cycle faces, we obtain an embedding of $K_{8,8,2}$ which is a minimum nonorientable genus embedding.

5 Families of embeddings

Transition graphs can be used to construct entire families of embeddings all at once, by making suitable pattern substitutions.

At left in Figure 3 we see a cyclic transition graph built from two type I patterns and ten type H patterns. The derived embedding is an embedding of $K_{12,12}$ with ten hamilton cycle faces and twelve 4-cycle faces. It is orientable because there are no twisted vertices in the transition graph. By inserting new vertices in the ten hamilton cycle faces, we obtain a minimum orientable genus embedding of $K_{12,12,10}$.

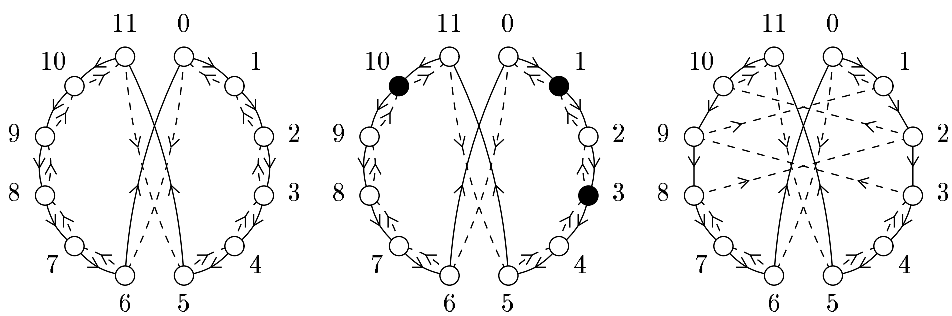

Figure 3

However, we can obtain other embeddings by making substitutions. In the middle of Figure 3 we have substituted three V patterns for six H patterns, thereby replacing six hamilton cycle faces by 36 4-cycle faces in the derived embedding of $K_{12,12}$. By adding vertices in the four remaining hamilton cycle faces, we obtain a minimum nonorientable genus embedding of $K_{12,12,4}$.

Also, at the right of Figure 3 we have substituted two X patterns for four H patterns, thereby replacing four hamilton cycle faces by 24 4-cycle faces in the derived embedding of $K_{12,12}$. By adding vertices in the six remaining hamilton cycle faces, we obtain a minimum orientable genus embedding of $K_{12,12,6}$.

These illustrate typical substitutions, replacing two H patterns at a time by a V pattern to construct families of nonorientable embeddings, or replacing four H patterns at a time by two X patterns, to construct families of orientable embeddings.

6 Relative embeddings and gadgets

Sometimes it is not possible to construct an embedding completely by algebraic means. We can use *partial transition graphs* to construct embeddings by a combination of algebraic and nonalgebraic techniques.

For example, suppose we wish to construct a minimum genus nonorientable embedding of $K_{9,9,4}$. Using transition graphs we might hope to construct an embedding of $K_{9,9}$ with four hamilton cycle faces into which we could insert four new vertices, and with all remaining faces being 4-cycles. Unfortunately, simple counting arguments show that four hamilton cycles together with an integral number of 4-cycles cannot cover every edge of $K_{9,9}$ twice, so this is impossible.

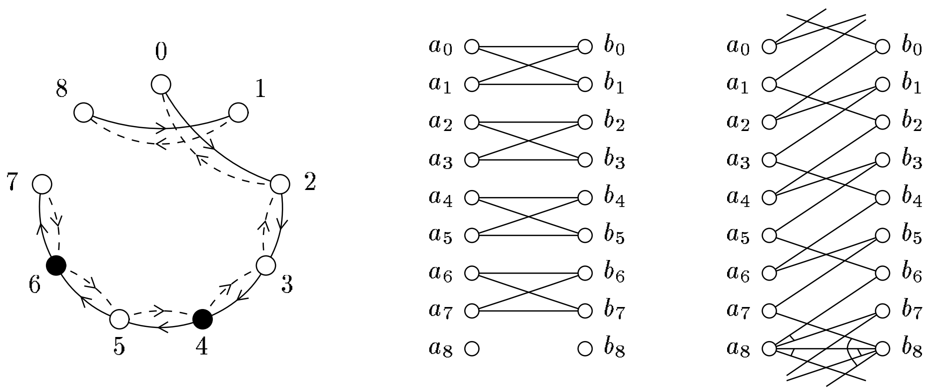

Figure 4

Instead, we use the partial transition graph shown at left in Figure 4 to construct a relative (or partial) embedding of $K_{9,9}$. We then complete this by adding the faces shown at middle and right in Figure 4, which we call a *gadget*. We then have an embedding with three hamilton cycle faces, one face that is a spanning closed walk of length 20, and with all remaining faces being 4-cycles. Adding new vertices in the hamilton cycle faces and the face of degree 20 gives a minimum nonorientable genus embedding of $K_{9,9,4}$.

7 Embeddings with finely tuned properties

Transition graphs make it possible to control the fine details of the derived embedding, in such a way that we may be able to perform further manipulations to obtain useful embeddings. One such manipulation is adding extra edges to the embedded graph without increasing the genus of the surface.

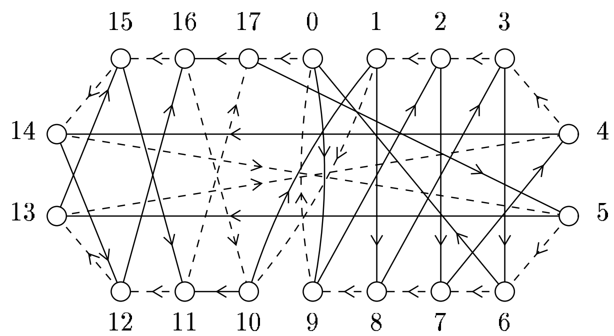

Figure 5

For example, in Figure 5 we see a cyclic transition graph made up of type I and X patterns only, that gives an orientable embedding of $K_{18,18}$ in which all faces are 4-cycles. This is then a minimum orientable genus embedding of $K_{18,18}$. This transition graph has been designed to have a special property: every possible solid edge *length* (voltage of head minus voltage of tail) between 1 and 17 occurs at least once. The existence of a solid edge of length k means that there is a face containing both b_i and b_{i+k} for every i. But for this transition graph k may take any nonzero value, so there is a 4-cycle face containing every pair of distinct vertices b_i and b_j. We may therefore add an edge joining every pair b_i and b_j, to obtain an embedding on the same surface of $\overline{K_{18}} + K_{18}$, the join of the empty graph $\overline{K_{18}}$ with the complete graph K_{18}. This is a minimum orientable genus embedding of that graph.

8 Applications

We have applied cyclic transition graphs, together with other techniques, to obtain the nonorientable genus of all complete tripartite graphs [3]. Two of us (Ellingham and Stephens) have also used cyclic transition graphs to obtain the orientable genus of $\overline{K_m}+K_n$ for n even and $m \geq n$ [2]. We have been working on determining the orientable genus of all complete tripartite graphs using cyclic transition graphs, and we have unpublished results for $K_{l,m,n}$, $l \geq m \geq n$, that resolve all cases except for three small graphs ($K_{12,11,8}$, $K_{13,11,8}$ and $K_{14,11,8}$) and the general case where $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$.

As we have shown, transition graphs are a useful and flexible tool for constructing graph embeddings. We hope that many other applications will be found for them in the future.

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