Drawing disconnected graphs on the Klein bottle

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How should someone simultaneously draw two distinct graphs on one surface. DeVos, Mohar and Samal conjectured the following in [1]:

Conjecture 1. Let G be the disjoint union of two graphs G_1 and G_2 and let Σ be a surface. In every optimal drawing of G on Σ the drawings of G_1 and G_2 are disjoint.

This conjecture is obviously true for the sphere, or equivalently for the Euclidean plane, and was announced as proven for the projective plane in [1]. In this work, we prove the following:

Theorem 1. *The conjecture holds if Σ is the Klein bottle.*

The problem remains open in the general case.

We denote by K the Klein bottle and by $\mathbb{C}_A$ and $\mathbb{C}_B$ its two crosscaps. A closed curve that does not bound an open disc in K is *essential.* A closed curve is one-sided if its tubular neighbourhood is a M?bius strip. According to [2], each essential non-separating simple closed curve in K belongs to one of the three following sets : the longitudes $\mathcal{C}_A$ which are one-sided curves isotopic to the boundary of $\mathbb{C}_A$, the longitudes $\mathcal{C}_B$ which are one-sided curves isotopic to the boundary of $\mathbb{C}_B$ and the meridians $\mathcal{C}_{AB}$ which are two-sided curves which cuts open K into an annulus.

To prove Theorem 1, we express the maximum number of edge-disjoint one-sided circuits in a drawing on K. This problem was first considered by Lins [3] in the projective plane.

Theorem 2. *Let Ψ be an embedding of an eulerian graph in the projective plane. Then the maximum number of pairwise edge-disjoint one-sided circuits equals the minimum of the cardinality of the intersection between a one-sided curve and Ψ.*

To establish a similar result in K, we use the following result of Schrijver:

Theorem 3. (Schrijver [5]) *Let Ψ be an embedding of an eulerian graph in the Klein bottle. Then the maximum number of pairwise edge-disjoint one-sided circuits equals the minimum number of the cardinality of the intersection between a one-sided curve edges intersecting every one-sided circuits.*

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Let Ψ denote a drawing of a graph G on K and $N(\Psi)$ the maximum number of edge-disjoint one-sided circuits in Ψ. We define k_a by

$$k_a = \min\{|\gamma \cap \Psi| \,|\, \gamma \in \mathcal{C}_A\}.$$

The numbers k_b and k_{ab} are defined similarly. Therefore, we can prove the key lemma :

Lemma 4. *Let G be an eulerian graph and Ψ an embedding of G in the Klein bottle, then $N(\Psi) = \min(k_a + k_b, k_{ab})$.*

We are then able to define some operations to draw graphs on smaller surfaces and have the Theorem holding for eulerian graphs. It then extends easily to every pair of graphs.

References

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