Superthrackles

Dan Archdeacon
Dept. of Math. and Stat.
University of Vermont
Burlington, VT, USA 05405
e-mail: dan.archdeacon@uvm.edu

Kirsten Stor
Dept. of Math. and Stat.
University of Vermont
Burlington, VT, USA 05405
e-mail: kstor@cems.uvm.edu

Extended Abstract

September 1, 2008

Summary: A drawing of a graph in the plane is a superthrackle if every edge crosses every other edge exactly once. We give a complete characterization of which graphs have a superthrackle drawing and explore related topics.

"Oh what a tangled web we weave, when first we practise to deceive!", Sir Walter Scott (1771-1832), *Marmion, Canto vi. Stanza 17.*

1 Introduction

A common way to represent a graph is to draw it in the plane. For clarity, the ideal is to draw it without crossings. Yet this is not always possible: Kuratowski's Theorem [3] says that a graph can be embedded in the plane if and only if it contains no subdivision of K_5 or of $K_{3,3}$. For non-planar graphs the goal is then to draw the graph in the plane with as few crossings as possible.

Let's make our concepts a bit more precise. A *drawing* of a graph $G = (V, E)$ in the plane is a subspace of the plane such that vertices V are represented by distinct points and edges E by curves (an image of $[0, 1]$) that join their incident endpoints and that contain no vertex in their interior (the image of $(0, 1)$). We further require that the image of any two distinct edges have a finite number of points in common, and if one of these common points lies in the interior of the edges, then the edges cross transversally at that point. Such a point will be called a *crossing*. We say that two edges *meet* at either a crossing point or at a vertex incident with both.

It is common to require three additional restrictions on drawings: namely,

- no edge crosses itself,
- two distinct edges cross at most once, and
- two edges incident with a common vertex do not cross.

A drawing that meets these additional three constraints is called *good*. Any drawing that minimizes the number of crossings is necessarily good.

Suppose that one were perverse and instead of looking for the *minimum* number of crossings in a good drawing one looked for the *maximum* number of crossings. A drawing where every pair of non-adjacent edges cross is called a *thrackle*. The word supposedly refers to a tangle of fishing line, and such drawings look very tangled indeed. A graph that can be drawn as a thrackle is *thrackleable*. In practice, we do not distinguish between a thrackle drawing and its underlying graph unless necessary to avoid confusion.

In the late 60's J.H. Conway posed the following conjecture. He offers a prize of 1000 in the currency of your choice for its solution or for a counterexample.

Conjecture 1.1 (Conway's Thrackle Conjecture) *The number of edges in thrackleable graph is no more than the number of vertices.*

This conjecture reduces to connected graphs. It is known that any tree is thrackleable, and that any cycle of length not equal to 4 is thrackleable (the latter result is a nice exercise for the reader). Connected graphs with a single cycle are called *unicycles.* D.R. Woodall [6] showed the following.

Theorem 1.1 *Given that Conway's Thrackle Conjecture is true, a graph is thrackleable if and only if it has at most one odd cycle, no cycle of length 4, and each component is a unicycle.*

Moreover, Woodall [6] showed that Conway's Thrackle Conjecture reduces to proving it for a special class of graphs.

Theorem 1.2 *Conways Thrackle Conjecture is true if and only if the one-point union of two even cycles is not thrackleable.*

The following [4] is a nice partial result on the Thrackle Conjecture. Let n refer to the order (number of vertices) of a graph.

Theorem 1.3 *If a graph is thrackleable, then the number of edges is at most* $2n - 3$.

This was later improved [1] to the following.

Theorem 1.4 *If a graph is thrackleable, then the number of edges is at most* $\frac{3}{2}(n-1)$.

We mention the following related result [4] and [5]. A definition of generalized thrackles is given in Section 3.

Theorem 1.5 *A bipartite graph can be drawn as a generalized thrackle if and only if it is planar.*

2 Superthrackles and our Main Result

We study a variation of a thrackle drawing. A *superthrackle drawing* is a drawing where *every* pair of edges cross exactly once. The distinction is that we require *adjacent* pairs of edges to cross, not just non-adjacent pairs as in a thrackle drawing. As above, a graph is *superthrackleable* if it has a superthrackle drawing. In the complete version of this paper we show:

Lemma 2.1 *If a graph is thrackleable, then it is superthrackleable.*

However, the converse is not true. Some graphs are superthrackleable, but not thrackleable.

Our main result is a complete characterization of graphs that are superthrackleable.

Theorem 2.1 (The Main Result) *A graph is superthrackleable if and only if it is either:*

- *bipartite and planar, or*
- *non-bipartite and projective-planar with all faces of even size.*

The rest of this section gives a sketch of the proof of Theorem 2.1. For details see the complete version of this paper. We break our sketch into 4 main statements, the first two covering the sufficiency of the two classes, and the second their necessity.

Statement 1: *If G is bipartite and planar, then it is superthrackleable.*

We first find a simple cycle C that crosses each edge exactly once. For convenience we redraw the graph with infinity representing one point on the cycle, and the remaining part of C as the x-axis. We then cut along the x-axis, shift the lower part downward and then revolve it around the y-axis. Reconnecting the edges using straight-line segments gives the desired drawing.

Statement 2: *If G is non-bipartite and projective-planar with all faces of even size, then it is superthrackleable.*

As before, we find a simple essential (non-contractible) cycle C that crosses each edge exactly once. The fact that such a cycle is essential follows from the fact that G is non-bipartite (a lemma in the complete paper). We redraw the graph in the plane outside of the unit circle, with the convention that a point p on the circle is identified with its antipode $-p$. We then consider this not as a circle with antipodes identified, but as a disk in the plane. Reconnecting the edges across this disk using straight-line segments gives the desired drawing.

Statement 3: *If G is bipartite and superthrackleable, then it is planar.*

The proof is a minor modification of the proof of Theorem 1.5 given in [4]. There are two main ideas that we outline here.

The first idea is to consider a *θ-graph*, that is, a homeomorph of $K_{2,3}$. A θ-graph has two vertices joined by three internally-vertex-disjoint paths p_1, p_2 and p_3. A drawing of a θ-graph is a *preserver* if the orientation of these three paths is the same at both degree-three vertices, and is called a *converter* otherwise. For example, a planar drawing of a θ-graph is a converter. Under the assumption that the drawing is a thrackle, or a superthrackle, one can use the parity of the length of the paths to determine if it must be a preserver or a converter.

The second idea is to consider the one-point union of two cycles, C_1 and C_2, at a vertex v. These cycles are *interlaced* in a drawing if they cross in a small neighborhood of v. Under the assumption that the drawing is a thrackle or a superthrackle, one can show that the cycles are interlaced if and only if both are of odd length.

Using these ideas the proof of Statement 3 follows as in [4].

Statement 4: *If G is non-bipartite and superthrackleable, then it is projective-planar with all faces of even size.*

We consider *parity embeddings* of G, that is, an embedding of G in a non-orientable surface such that even-length cycles are orientation-preserving and odd-length cycles are orientation-reversing. More generally, a *signed graph* $G^{\pm}$ is a graph with a plus or minus sign on each edge. This signature induces a signature on each cycle. A *signed embedding* of $G^{\pm}$ is an embedding of G where positively-signed cycles are orientation-preserving and negatively-signed cycles are orientation-reversing. A parity embedding of G is a signed embedding of G^{-}, the signed graph with every edge negative. Two signed graphs are *switching equivalent* if there is a subset A of vertices such that the sign on an edge e changes if and only if e is incident with an odd number of vertices in A. Finally, an *even-subdivision* of a signed graph $G^{\pm}$ is a graph where some subset of the edges has been subdivided an even number of times, every new edge receiving a positive sign. The signature on a cycle is unchanged in an even subdivision.

Zaslavsky [7] has given a forbidden signed-subgraph theorem characterizing those signed graphs that embed in the projective plane. As a corollary, this characterizes those graphs with a parity embedding in the projective plane.

Theorem 2.2 *A signed graph embeds in the projective plane if and only if it* *contains no signed subdivision that is switching equivalent to one of 8 specific graphs.*

Corollary 2.1 *A graph G has a parity embedding in the projective plane if and only if it contains no even subdivision of one of 8 specific graphs.*

To prove Statement 4 we show that no even subdivision of one of Zaslavsky's 8 excluded subgraphs is superthrackleable. Equivalently, if a graph is superthrackleable, then it has a parity embedding in the projective plane. Since any face is orientation-preserving and G is assumed to be non-bipartite, the statement follows.

Final Comments: We end this section by observing that Statements 1–4 combine to prove Theorem 2.1. The construction techniques of Statements 1 and 2 also show:

Corollary 2.2 *If G is superthrackleable, then there is a superthrackle drawing where all crossings occur at the same point.*

3 Generalized Thrackles

Other variations on a thrackle drawing have been studied. A *generalized thrackle* is a drawing where each edge meets each other edge an odd number of times. Not counting incidences, this says that two non-adjacent edges cross an odd number of times, while two adjacent edges cross an even number of times. Cairns and Nikolayevsky [2] characterized graphs that are generalized-thrackleable.

Theorem 3.1 *A graph is generalized-thrackleable if and only if it is either:*

- *bipartite and planar, or*
- *non-bipartite and projective-planar with all faces of even size.*

The relationship with our main result is striking.

Corollary 3.1 *A graph is superthrackleable if and only if it is generalized-thrackleable.*

The corollary follows since these two classes of graphs have exactly the same characterization. One might hope for a more direct proof. The best we could do is:

Lemma 3.1 *If a graph is superthrackleable, then it is generalized-thrackleable.*

The converse appears difficult without using their common characterization. In part, this is because in a generalized-thrackle non-adjacent edges are allowed to cross an odd number of times, but in a superthrackle they can cross only once. The work of Pelsmajer, Schaefer and Stefankovic may be helpful here.

N.B. This work is part of the second author's PhD research.

References

[1] G. Cairns and Y. Nikolayevsky, Bounds for generalized thrackles, *Discrete Comput. Geom.*, **23** (2000),191-206.

[2] G. Cairns and Y. Nikolayevsky, Generalized thrackle drawings of non-bipartite graphs, preprint.

[3] K. Kuratowski, Sur le probleme de courbes gauches en topologie, *Fund. Math.* **15** (1930) 271–283.

[4] L. Lovasz, J. Pach, and M. Szegedy, On Conway's thrackle conjecture, *Discrete Comput. Geom.*, **18** (1997),369-376.

[5] M.J. Pelsmajer, M. Schaefer, and D. Stefankovic, *Electronic Notes in Discrete Mathematics* Volume **29**, 15 August 2007, Pages 85-90. European Conference on Combinatorics, Graph Theory and Applications.

[6] D.R. Woodall, Thrackles and deadlock, in *Combinatorial mathematics and its applications* (D. Welsh ed.), Academic Press, New York, 1971, pp. 335–347.

[7] T. Zaslavsky, The projective-planar signed graphs, *Discrete Mathematics* **113** (1993) 233–247.